Teacher notes
Topic A

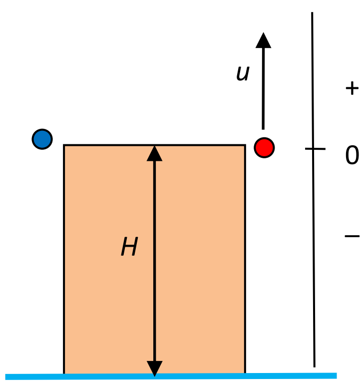
A peculiar problem for motion under gravity

In this problem we will take $g = 10\ \text{m s}^{-2}$.

A ball is projected vertically upwards with an initial speed *u* from the edge of a cliff at a height *H* above sea level. At a time of $\tau = 1\ \text{s}$ later an identical ball is dropped from the edge of the cliff. Both balls reach the sea at the same time.

What is *H*?

What are the possible values of *u*?



For the red ball: it reaches the ground at time *t* given by $-H = ut - \frac{1}{2}gt^2$.

For the blue ball: it reaches the ground at time *t* given by $-H = -\frac{1}{2}g(t-\tau)^2$.

Hence

$$ut-\frac{1}{2}gt^2=-\frac{1}{2}g(t-\tau)^2$$

$$ut-\frac{1}{2}gt^2=-\frac{1}{2}gt^2+gt\tau-\frac{g\tau^2}{2}$$

$$ut=+gt\tau-\frac{g\tau^2}{2}$$

$$t=\frac{1}{2}\frac{g\tau^2}{g\tau-u}$$

This implies that

$$-H=ut-\frac{1}{2}gt^2$$

$$H=-u\times\frac{1}{2}\frac{g\tau^2}{g\tau-u}+\frac{1}{2}g(\frac{1}{2}\times\frac{g\tau^2}{g\tau-u})^2$$

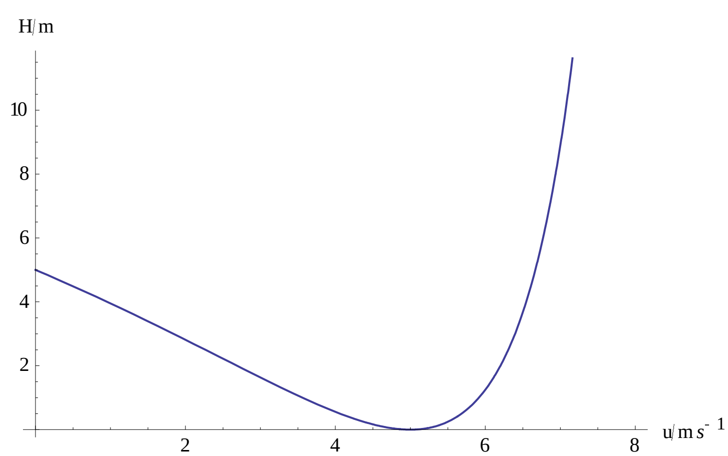
$$H=\frac{g^3\tau^4-4ug\tau^2(g\tau-u)}{8(g\tau-u)^2}=\frac{g^3\tau^4-4ug^2\tau^3+4u^2g\tau^2}{8(g\tau-u)^2}=\frac{g\tau^2(g^2\tau^2-4ug\tau+4u^2)}{8(g\tau-u)^2}$$

$$H=\frac{g\tau^2(g\tau-2u)^2}{8(g\tau-u)^2}$$

So we have: time to hit the ground is $t=\frac{1}{2}\frac{g\tau^2}{g\tau-u}$ and the height is given by $H=\frac{g\tau^2(g\tau-2u)^2}{8(g\tau-u)^2}$.

But, the time has to be positive and so $g\tau-u>0$ which implies $u<g\tau$. There is a maximum speed involved, $u_{max}=g\tau$. There is also a minimum speed involved: if the launch speed *u* is too low the ball will come back to its initial position in a time less than τ seconds and so the two balls cannot reach the ground at the same time. This means $u>\frac{g\tau}{2}$.

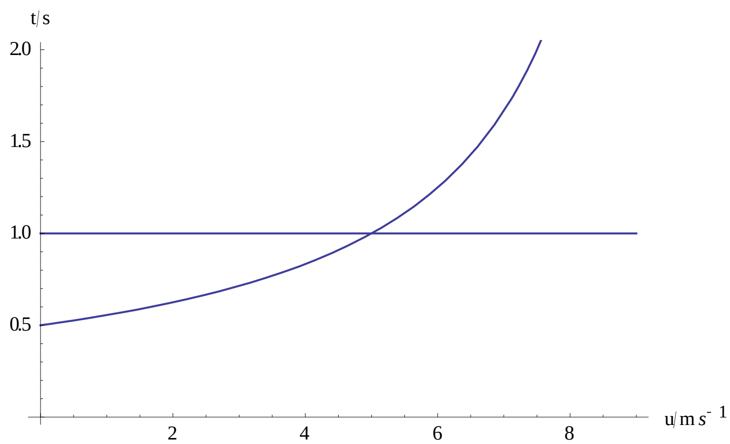
For $\tau=1$ s we have the following graphs of *H* versus *u* and *t* versus *u*:



The height gets bigger and bigger as the speed approaches $u_{max} = g\tau = 10\ \text{ms}^{-1}$.

The height has to be 0 when the speed approaches the minimum of $u_{min} = \dfrac{g\tau}{2} = 5.0\ \text{ms}^{-1}$.

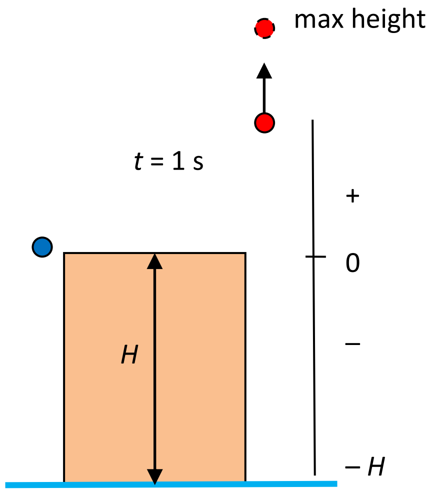
For any speed between u_{min} and u_{max} the height is given by the formula we derived above.



To see what happens if the launch speed u is outside these limits consider the cases $u = 2.0\ \text{ms}^{-1}$ and $u = 20\ \text{ms}^{-1}$.

For $u = 2.0\ \text{ms}^{-1}$ the red ball reaches maximum height at 0.2 s and returns to its initial place at 0.4 s. It then continues towards the sea before the blue ball starts to move. The two balls will not reach the sea at the same time no matter what *H* is.

For $u = 20\ \text{ms}^{-1}$ the red ball reaches maximum height at 2 s and returns to its initial place at 4 s. The blue ball starts moving *before* the red ball reaches max height.



There is no way they can reach the sea at the same time no matter what the height *H* is.